Oct. 2010 CHINESE JOURNAL OF ENGINEERING MATHEMATICS Vol. 27 No. 5

**Article ID:**1005-3085(2010)05-0918-07

# A New Method Based on Reproducing Kernel Space for Boundary Value Problems

LI Jian[1,2]

(1- Department of Mathematics, Harbin Institute of Technology, Harbin 150001;

2- Operation Research Center, Nanjing Army Command College, Nanjing 210045)

**Abstract:** In this paper, we present a new method for solving singularly perturbed two-point singular boundary value problems. Its exact solution is represented in a form of series in reproducing kernel space. In the mean time, the $n$-term approximation $u_n(x)$ to the exact solution $u(x)$ is obtained and is proved to converge to the exact solution. Some numerical examples are studied to demonstrate the accuracy of the new method. Results indicate that the new method is simple and effective.

**Keywords:** exact solution; singularly perturbed two-point singular boundary value problem; reproducing kernel

**Classification:** AMS(2000) 37M20 **CLC number:** O241.81 **Document code:** A

## 1 Introduction

In this paper, we consider the following singularly perturbed two-point singular boundary value problem in reproducing kernel space

$$\begin{cases} \varepsilon u''(x) + \dfrac{k}{p(x)}u'(x) + \dfrac{s}{q(x)}u(x) = f(x), \quad 0 < x \leq 1, \\ u(0) = 0, \\ u(1) = 0, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$, $u(x) \in W_2^3[0,1]$, $p(x) = O(x^\alpha)\,(\alpha > 0)$ and $q(x) = O(x^\beta)\,(\beta > 0)$.

The singularly perturbed differential equations arise in a variety of differential applied mathematics, fluid mechanics, quantum mechanics, optimal control, gas dynamics, nuclear physics, chemical reaction, studies of atomic structures and atomic calculations. Therefore, the problem has attracted much attention and has been studied by many authors. In general, classical numerical methods fail to produce good approximations for these equations. Hence one has to go for non-classical method. Some non-classical methods have been suggested by various authors[1-4]. But only few authors have developed numerical methods for singularly perturbed two-point singular boundary value problem. Mohanty and his co-workers have given the numerical solution of singularly perturbed two-point singular boundary value problem using convergent tension spline method and non-uniform mesh tension spline method[5,6].

**Received:** 02 Apr 2008. **Accepted:** 08 Sep 2009.

**Biography**: Li Jian (Born in 1982), Male, Master. Research field: kernel methods and data mining.

In this paper, we will give a representation of exact solution to (1) and approximate solution in the reproducing kernel space under the assumption that the solution to (1) is unique. The approach is simple and effective.

For convenience, we take $p(x) = x$ and $q(x) = x^2$ in (1). After multiplying (1) by $x^2$, we find that

$$\begin{cases} \varepsilon x^2 u''(x) + xu'(x) + u(x) = x^2 f(x), \quad 0 \le x \le 1, \\ u(0) = 0, \\ u(1) = 0, \end{cases} \tag{2}$$

where $0 < \varepsilon \ll 1$, $u \in W_2^3[0,1]$, $x^2 f(x) \in W_2^1[0,1]$.

Clearly, the solution to (2) is the solution to (1). So we only need to obtain the solution of (2). Write $F(x) = x^2 f(x)$ simply and put $Lu \equiv x^2 u''(x) + xu'(x) + u(x)$. Then (1) can further be converted into the following form:

$$\begin{cases} Lu(x) = F(x), \quad 0 \le x \le 1, \\ u(0) = 0, \\ u(1) = 0, \end{cases} \tag{3}$$

where $0 < \varepsilon \ll 1$, $u \in W_2^3[0,1]$, $F(x) \in W_2^1[0,1]$. $W_2^1[0,1]$ and $W_2^3[0,1]$ are defined in the following section.

## 2 Several reproducing kernel spaces

### 2.1 The reproducing kernel space $W_2^3[0,1]$

The inner product space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{u(x) \mid u, u', u''$ are absolutely continuous real value functions, $u$, $u'$, $u''$, $u^{(3)} \in L^2[0,1]$, $u(0) = 0$, $u(1) = 0\}$. The inner product in $W_2^3[0,1]$ is given by

$$\big(u(y), v(y)\big)_{W_2^3} = \int_0^1 \big(36uv + 49u'v' + 14u''v'' + u^{(3)}v^{(3)}\big)\mathrm{d}y, \tag{4}$$

and the norm $\|u\|_{W_2^3}$ is denoted by $\|u\|_{W_2^3} = \sqrt{(u,u)_{W_2^3}}$, where $u$, $v \in W_2^3[0,1]$.

**Theorem 2.1** The space $W_2^3[0,1]$ is a reproducing kernel space. That is, for any $u(y) \in W_2^3[0,1]$ and each fixed $x \in [0,1]$, there exists $R_x(y) \in W_2^3[0,1]$, $y \in [0,1]$, such that

$$\big(u(y), R_x(y)\big)_{W_2^3} = u(x).$$

The reproducing kernel $R_x(y)$ can be denoted by

$$R_x(y) = \begin{cases} c_1 e^{y} + c_2 e^{-y} + c_3 e^{2y} + c_4 e^{-2y} + c_5 e^{3y} + c_6 e^{-3y}, \quad y \le x, \\ d_1 e^{y} + d_2 e^{-y} + d_3 e^{2y} + d_4 e^{-2y} + d_5 e^{3y} + d_6 e^{-3y}, \quad y > x. \end{cases} \tag{5}$$

The coefficients of the reproducing kernel $R_x(y)$ and the proof of Theorem 2.1 are given in [8].

### 2.2 The reproducing kernel space $W_2^1[0,1]$

The inner product space $W_2^1[0,1]$ is defined by $W_2^1[0,1] = \{u(x) \mid u$ is absolutely continuous real value function, $u, u' \in L^2[0,1]\}$. The inner product and norm in $W_2^1[0,1]$ are given, respectively, by

$$\big(u(x), v(x)\big)_{W_2^1} = \int_0^1 (uv + u'v')\mathrm{d}x, \quad \|u\|_{W_2^1} = \sqrt{(u,u)_{W_2^1}},$$

where $u(x)$, $v(x) \in W_2^1[0,1]$. In [7], the authors have proved that $W_2^1[0,1]$ is a complete reproducing kernel space and its reproducing kernel is

$$\overline{R}_x(y) = \frac{1}{2\sinh(1)}\big[\cosh(x+y-1) + \cosh\big(|x-y|-1\big)\big].$$

## 3 The solution of (3)

In this section, the solution of (3) is given in the reproducing kernel space $W_2^3[0,1]$.

In (3), it is clear that $L: W_2^3[0,1] \to W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{R}_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$ where $L^*$ is the adjoint operator of $L$. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^3[0,1]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad \beta_{ik} > 0, \quad i = 1, 2, \cdots. \tag{6}$$

**Theorem 3.1** For (3), if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$ and $\psi_i(x) = L_y R_x(y)|_{y=x_i}$.

**Proof** We have

$$\psi_i(x) = (L^*\varphi_i)(x) = \big((L^*\varphi_i)(y), R_x(y)\big) = \big(\varphi_i(y), L_y R_x(y)\big) = L_y R_x(y)|_{y=x_i}.$$

The subscript $y$ by the operator $L$ indicates that the operator $L$ applies to the function of $y$.

Clearly, $\psi_i(x) \in W_2^3[0,1]$. For each fixed $u(x) \in W_2^3[0,1]$, let $(u(x), \psi_i(x)) = 0$, $i = 1, 2, \cdots$, which means that

$$\big(u(x), (L^*\varphi_i)(x)\big) = \big(Lu(\cdot), \varphi_i(\cdot)\big) = (Lu)(x_i) = 0. \tag{7}$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$. Hence, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of $L^{-1}$. So the proof of Theorem 3.1 is completed.

**Theorem 3.2** If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and the solution of (3) is unique, then the solution of (3) is

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x). \tag{8}$$

**Proof** Applying Theorem 3.1, it is easy to know that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^3[0,1]$.

Noting that $(v(x), \varphi_i(x)) = v(x_i)$ for each $v(x) \in W_2^1[0,1]$, we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} (u(x), \overline{\psi}_i(x))\overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(u(x), L^*\varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(Lu(x), \varphi_k(x))\overline{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(f(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x). \end{aligned} \tag{9}$$

So the proof of Theorem 3.2 is completed.

Now, the approximate solution $u_n(x)$ can be obtained by the $n$-term intercept of the exact solution $u(x)$ and

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x). \tag{10}$$

**Theorem 3.3** Assume $u(x)$ is the solution of (3) and $r_n(x)$ is the error between the approximate $u_n(x)$ and the exact solution $u(x)$. Then the error $r_n(x)$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^3}$.

**Proof** From (9), (10), it follows that

$$\begin{aligned} \| r_n \|_{W_2^3} &= \left\| \sum_{i=n+1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x) \right\|_{W_2^3} \\ &= \sum_{i=n+1}^{\infty} \Big( \sum_{k=1}^{i} \beta_{ik} f(x_k) \Big)^2. \end{aligned} \tag{11}$$

(11) shows that the error $r_n$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^3}$. So the proof of Theorem 3.3 is completed.

## 4 Numerical example

In this section, some numerical examples are studied to demonstrate the accuracy of the presented method. The examples are computed using Mathematica 4.2. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

**Example 1** Consider the equation

$$\begin{cases} \varepsilon u''(x) + \dfrac{1}{x}u'(x) + \dfrac{1}{x^2}u(x) = f(x), \quad 0 < x \le 1, \\ u(0) = 0, \\ u(1) = 0, \end{cases}$$

where $f(x) = \frac{2}{x} - 2\varepsilon - 3$. The true solution is $x - x^2$. Using our method, we choose 26 points on $[0, 1]$ and take $\varepsilon = 10^{-2}$, $10^{-6}$, $10^{-10}$, respectively. The numerical results are given in Table 1, Table 2 and Table 3.

*Table 1: Numerical results for Example 1* ($n=26$, $\varepsilon=10^{-2}$)

| $x$ | true solution $u(x)$ | approximate solution $u_{26}$ | absolute error |
|---|---|---|---|
| 0.0001 | 9.999E-05 | 9.99901E-05 | 1.2E-10 |
| 0.08 | 0.0736 | 0.0736 | 3.7E-11 |
| 0.16 | 0.1344 | 0.1344 | 8.3E-09 |
| 0.32 | 0.2176 | 0.2176 | 3.5E-08 |
| 0.48 | 0.2496 | 0.2496 | 8.0E-08 |
| 0.64 | 0.2304 | 0.2306 | 1.5E-07 |
| 0.80 | 0.1600 | 0.1600 | 2.4E-07 |
| 0.96 | 0.0384 | 0.0383996 | 4.4E-07 |
| 1.00 | 0 | 0 | 0 |

*Table 2: Numerical results for Example 1* ($n=26$, $\varepsilon=10^{-6}$)

| $x$ | true solution $u(x)$ | approximate solution $u_{26}$ | absolute error |
|---|---|---|---|
| 0.0001 | 9.999E-05 | 9.99901E-05 | 1.2E-10 |
| 0.08 | 0.0736 | 0.0736 | 1.1E-09 |
| 0.16 | 0.1344 | 0.1344 | 4.1E-09 |
| 0.32 | 0.2176 | 0.2176 | 1.4E-08 |
| 0.48 | 0.2496 | 0.2496 | 2.2E-08 |
| 0.64 | 0.2304 | 0.2306 | 2.5E-08 |
| 0.80 | 0.1600 | 0.1600 | 2.1E-08 |
| 0.96 | 0.0384 | 0.0384 | 1.2E-08 |
| 1.00 | 0 | 0 | 0 |

*Table 3: Numerical results for Example 1* ($n=26$, $\varepsilon=10^{-10}$)

| $x$ | true solution $u(x)$ | approximate solution $u_{26}$ | absolute error |
|---|---|---|---|
| 0.0001 | 9.999E-05 | 9.99901E-05 | 1.2E-10 |
| 0.08 | 0.0736 | 0.0736 | 1.1E-09 |
| 0.16 | 0.1344 | 0.1344 | 4.1E-09 |
| 0.32 | 0.2176 | 0.2176 | 1.4E-08 |
| 0.48 | 0.2496 | 0.2496 | 2.2E-08 |
| 0.64 | 0.2304 | 0.2306 | 2.5E-08 |
| 0.80 | 0.1600 | 0.1600 | 2.1E-08 |
| 0.96 | 0.0384 | 0.0384 | 1.2E-08 |
| 1.00 | 0 | 0 | 0 |

**Example 2** Consider the equation

$$\begin{cases} \varepsilon u''(x) + \dfrac{1}{x \sin x} u'(x) + \dfrac{1}{x^2} u(x) = f(x), & 0 < x \leq 1, \\ u(0) = 0, \\ u(1) = 0, \end{cases}$$

where

$$f(x) = \frac{\pi \cos(\pi x) \csc x}{x} - \varepsilon \pi^2 \sin(\pi x) + \frac{\sin(\pi x)}{x^2}.$$

The true solution is $\sin(\pi x)$. Using our method, we choose 26 points on $[0, 1]$ and take $\varepsilon = 10^{-2}$, $10^{-6}$, $10^{-10}$, respectively. The numerical results are given in the following Table 4, Table 5 and Table 6.

*Table 4: Numerical results for Example 2* ($n = 26$, $\varepsilon = 10^{-2}$)

| $x$ | true solution $u(x)$ | approximate solution $u_{25}$ | absolute error |
|---|---|---|---|
| 0.0001 | 0.0314159 | 0.0314157 | 1.8E-09 |
| 0.08 | 0.24869 | 0.248643 | 4.6E-05 |
| 0.16 | 0.481754 | 0.481702 | 5.1E-05 |
| 0.32 | 0.844328 | 0.844275 | 5.2E-05 |
| 0.48 | 0.998027 | 0.997974 | 5.2E-05 |
| 0.64 | 0.904827 | 0.904776 | 5.1E-05 |
| 0.80 | 0.587785 | 0.587736 | 4.9E-05 |
| 0.96 | 0.125333 | 0.125274 | 5.9E-05 |
| 1.00 | 0 | 0 | 0 |

*Table 5: Numerical results for Example 2* ($n = 26$, $\varepsilon = 10^{-6}$)

| $x$ | true solution $u(x)$ | approximate solution $u_{25}$ | absolute error |
|---|---|---|---|
| 0.0001 | 0.0314159 | 0.0314157 | 1.8E-09 |
| 0.08 | 0.24869 | 0.248643 | 4.6E-05 |
| 0.16 | 0.481754 | 0.481702 | 5.1E-05 |
| 0.32 | 0.844328 | 0.844275 | 5.2E-05 |
| 0.48 | 0.998027 | 0.997974 | 5.2E-05 |
| 0.64 | 0.904827 | 0.904776 | 5.1E-05 |
| 0.80 | 0.587785 | 0.587736 | 4.9E-05 |
| 0.96 | 0.125333 | 0.125274 | 5.9E-05 |
| 1.00 | 0 | 0 | 0 |

*Table 6: Numerical results for Example 2* ($n = 26$, $\varepsilon = 10^{-10}$)

| $x$ | true solution $u(x)$ | approximate solution $u_{25}$ | absolute error |
|---|---|---|---|
| 0.0001 | 0.0314159 | 0.0314157 | 1.8E-09 |
| 0.08 | 0.24869 | 0.248643 | 4.6E-05 |
| 0.16 | 0.481754 | 0.481702 | 5.1E-05 |
| 0.32 | 0.844328 | 0.844275 | 5.2E-05 |
| 0.48 | 0.998027 | 0.997974 | 5.2E-05 |
| 0.64 | 0.904827 | 0.904776 | 5.1E-05 |
| 0.80 | 0.587785 | 0.587736 | 4.9E-05 |
| 0.96 | 0.125333 | 0.125274 | 5.9E-05 |
| 1.00 | 0 | 0 | 0 |

## References:

[1] Miller J J H, Riordan E O, Shishkin G I. On piecewise uniform meshs for upwind and central differential operators for solving singular perturbed problems[J]. SIMA J Numer Anal, 1995, 15: 89-99

[2] Kreiss B, Kreiss H O. Numerical methods for singularly perturbation problem[J]. SIAM J Numer Anal, 1982, 46: 138-165

[3] Yekaterinburg. Asymptotic expansion of a solution of a singularly perturbed system of linear equation[J]. Differential Equation, 2005, 41(10): 1392-1402

[4] Ilicasu F O, Schultz D H. High-order finite-difference techniques for linear singular perturbation boundary value problems[J]. Computers and Mathematics with Applications, 2004, 47: 391-417

[5] Mohanty R K, Evans D J, Arora U. Convergent spline in tension methods for singularly perturbed two point singular boundary value problems[J]. Int J Comput Math, 2005, 82: 55-66

[6] Mohanty R K, Urvashi Arora. A family of non-uniform mesh tension spline methods for singularly perturbed two-point singular boundary value problems with significant first derivatives[J]. Applied Mathematics and Computation, 2006, 172: 531-544

[7] Li C L, Cui M G. The exact solution for solving a class nonlinear operator equations in the reproducing kernel space[J]. Appl Math Compu, 2003, 143(2-3): 393-399

[8] Geng F Z, Cui M G. Solving a nonlinear system of second order boundary value problems[J]. Journal of Mathematical Analysis and Applications, 2007, 327: 1167-1181

# 一种基于再生核空间解决边值问题的新方法

李 健[1,2]

(1- 哈尔滨工业大学理学院数学系，哈尔滨 150001; 2- 南京陆军指挥学院作战实验中心，南京 210045)

**摘 要**: 在本论文中，我们介绍一种解决奇异摄动两点边值问题新的方法。它的精确解在再生核空间中是以级数形式出现的。同时，存在 $n$ 项解 $u_n(x)$ 近似于精确解 $u(x)$，并且可以证明 $u_n(x)$ 收敛于 $u(x)$。我们在此举出一些数列来验证本方法的精确性，用此方法得出的解，显示了这种方法的简单有效。

**关键词**: 精确解；奇异摄动两点边值问题；再生核